B.Sc. part – I (Math)
paper – II
The cone (problem)

Q-1 prove that the equation
$x^2-2y^2+3z^2-4xy+5yz-6zx+8x-19y$
$-2z-20=0$
represents a cone. Find the co-ordinate of its vertex.

Soln:– The given eqn. is
$x^2-2y^2+3z^2-4xy+5yz-6zx+8x-19y$
$-2z-20=0$ —— (1)

Let (α,β,γ) be such a point that when the origin is shifted (α,β,γ) the eqn (1) becomes homogeneous then the equation (1) will represent a cone.

Now shifting the origin to (α,β,γ) the eqn (1) becomes
$(x+\alpha)^2-2(y+\beta)^2+3(z+\gamma)^2-4(x+\alpha)(y+\beta)$
$+5(y+\beta)(z+\gamma)-6(z+\gamma)(x+\alpha)+$
$8(x+\alpha)-19(y+\beta)-2(z+\gamma)-20=0$

$x^2-2y^2+3z^2-4xy+5yz-6zx+$
$(2\alpha-4\beta-6\gamma+8)x+(-4\beta-4\alpha+5\gamma-19)y$
$+(6\gamma+5\beta-6\alpha-2)z+(\alpha^2-2\beta^2+3\gamma^2$
$-4\alpha\beta+5\beta\gamma-6\gamma\alpha+8\alpha-19\beta-2\gamma-20)=0$

If it is represent a cone we must have
$2\alpha-4\beta-6\gamma+8=0$ —— (2)

$-4\alpha - 6\beta + 5\gamma - 19 = 0$ — ③

$-6\alpha + 5\beta + 6\gamma - 2 = 0$

and ϕ $\alpha^2 - 2\beta^2 + 3\gamma^2 - 4\alpha\beta + 5\beta\gamma + 6\gamma\alpha$
$+ 8\alpha - 19\beta - 2\gamma - 20 = 0$ — ⑧

solving eqn ② ③ ④ we get

$\alpha = 1, \beta = -2, \gamma = 3$

This eqn ⑤ is also satisfied for these value of α, β, γ and homogeneous when the origin shifted to $(1, -2, 3)$

Thus eqn. represents a cone whose vertex is $(1, -2, 3)$.

Q: — prove that the plane $ax + by + c$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$

Soln:— Here the sum of the co-efficient of x^2, y^2, z^2 in the equation of the cone is equal to zero. Therefore it is a cone which has three mutually perpendicular generators. Hence the plane $ax + by + cz = 0$ will cut the cone in perpendicular generator of the normal to the plane $ax+by+cz=0$ through the vertex is a generator of cone. Now the equation of the line normal to $ax+by+cz=0$ and passing through the vertex is

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$$

This will be generator of the cone if $bc+ca+ab=0$

diving by abc we get

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$$

Q. — Find the equation of the cone whose vertex is the point (α, β, γ) and whose generating lines pass through the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

Soln: — The eqn of line passing through (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{— (1)}$$

$\therefore x = \alpha - \frac{l\gamma}{n}, \; y = \beta - \frac{m\gamma}{n}$

The point of intersection of line ① and the plane $z=0$ is $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

If this lies on the conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ then

$$\frac{1}{a^2}\left(\alpha - \frac{l\gamma}{n}\right)^2 + \frac{1}{b^2}\left(\beta - \frac{m\gamma}{n}\right)^2 = 1$$

Now eliminating l, m, n from ① and ② we get

$$\frac{1}{a^2}\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)^2 + \frac{1}{b^2}\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)^2 = 1$$

or $\frac{1}{a^2}\left[\alpha(z-\gamma) - \gamma(x-\alpha)\right]^2 + \frac{1}{b^2}\left[\beta(z-\gamma) - \gamma(y-\beta)\right]^2 = (z-\gamma)^2$

or $\frac{1}{a^2}(\alpha z - \gamma x)^2 + \frac{1}{b^2}(\beta z - \gamma y)^2 = (z-\gamma)^2$

or $b^2(\alpha z - \gamma x)^2 + a^2(\beta z - \gamma y)^2 = a^2 b^2 (z-\gamma)^2$

This is required eqn. of cone